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PLURALITY AND PROBABILITY OF VICTORY: SOME EQUIVALENCE RESULTS

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Plurality and Probability of Victory: Some Equivalence Results

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Abstract

This paper examines decision-making by political candidates under three different objective functions. In particular, we are interested in when the optimal strategies for expected vote share, expected plurality, and probability of victory maximizing candidates coincide in simple plurality elections. It is shown here that if voters' behavior, conditional on the policies proposed by the candidates, is identical from the candidates' perspective, and candidates are restricted to choosing pure strategies, then all three objectives lead to the same best response function when there are two candidates and abstention is not allowed. We then provide a counter-example to Hinich's claim of general asymptotic equivalence in two candidate elections without abstention in which voter types are independently, but not identically distributed. In addition, we provide a counterexample to general best response equivalence between these objective functions in two candidate elections in which abstention is allowed, but our other assumptions are satisfied. Finally, an example of why our result can not be immediately extended to arbitrary numbers of candidates is provided.

1 Introduction

Spatial models of elections often assume that the candidates' sole goal is victory. To calculate the optimal strategy for such a candidate, one must take into account the probability of victory resulting from each strategy. In general, this probability is not a trivial computation, especially when studying probabilistic voting models (e.g. Hinich (1977), Coughlin and Nitzan (1981a), (1981b), Ledyard (1984), and McKelvey and Patty

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(1999)). For this reason, researchers have sought candidate objectives which are easier to compute and yet lead to the same predictions as those generated by probability of victory.

If two candidate objective functions yield identical best response functions, we will refer to them as being *equivalent*. Regardless of one's opponents' strategies, the optimal strategy under one objective function is identical to that under the other objective function when the two are equivalent.

It follows immediately that equivalent objective functions yield identical Nash equilibrium predictions.[1] Two objective functions may, on the other hand, still yield identical Nash equilibrium strategies without being equivalent. We refer to such objective functions as exhibiting *equilibrium equivalence*. Either type of equivalence is useful, since theoretical treatments of candidate competition are often concerned solely with Nash equilibria, but we will limit ourselves to the question of best-response equivalence in this paper. The question of equilibrium equivalence between maximization of expected vote share and maximization of probability of victory is examined in Patty (1999).

Several articles discussing properties of different candidate strategies were published in the 1970s. Foremost among these early efforts is Aranson, Hinich, and Ordeshook (1974). Aranson, *et al.* offer an equivalence result which rests on assumptions regarding perturbations of the candidate's objective functions, perhaps representing forecast errors. Their result, however, requires that these forecast errors are unbiased and, more importantly, that the errors are uncorrelated with the strategies chosen by the candidates. As the authors point out, this assumption is untenable, since the value of the objective functions (even after the errors are taken into account) must fall between zero and one. A second equivalence result obtained by Aranson, *et al.* requires that the votes received in a two candidate election be distributed according to a multivariate normal distribution. This obviously requires that negative vote totals be a positive probability event. Aranson, *et al.* were unable to offer any equivalence results between expected plurality and probability of victory which did not depend on *ad hoc* assumptions, however.

Hinich (1977), however, provided justification for examining expected vote share in place of probability of victory which depended only on the Central Limit Theorem. Asymptotically, Hinich claimed, the two objective functions converged in 2 candidate elections without abstention. This claim was extended by Ledyard (1984) to include 2 candidate elections in which abstention is allowed. Neither Hinich nor Ledyard proved their results, each for different reasons. Hinich's proposition is merely a statement. Ledyard's paper does not need a general equivalence result for the major conclusions to hold.[2]

The existing literature has shown that equivalence results do not hold in general.

[1] For a more detailed discussion of this, see Aranson, Hinich, and Ordeshook (1974), p. 144-145.

[2] Ledyard argues at the limit, which is never actually realized in his framework. In addition, there is a discontinuity at the limit, making his argument impossible to generalize immediately for finite numbers of voters.

In order to show equivalence between maximizing plurality and probability of victory, three things are assumed to hold. Voters do not condition their actions on the actions of other voters, voters are using symmetric voting strategies and candidate's strategies are announced simultaneously. In Section 3, a statement and discussion of Hinich's claim is given, as well as a statement of Ledyard's result. The focus of the section is Theorem 1, which states that Hinich's claim holds for any number of voters so long as the assumptions imposed in this paper hold. In Section 4 abstention is allowed with two candidates. It is shown that Theorem 1 can not be extended to this case. Section 5 concludes with three examples and offers possible extensions of this work.

2 The Model

Let $\mathcal{N}$ denote a finite set of voters, with $|\mathcal{N}| = N$, and $\mathcal{J}$ denote a finite set of alternatives, with $|\mathcal{J}| = J$. We denote the set of *candidates* by $\mathcal{J}^0 \subset \mathcal{J}$, with $|\mathcal{J}^0| = J^0$. Each candidate j chooses a *policy*, $x_j \in X$, where X denotes the policy space. We will denote the J^0-dimensional vector of all policies by x, and the space of all such vectors by $Y = X^{J^0}$. Finally, we will denote the vector of all announced policies, other than the policy announced by candidate j, by x_{-j}.

2.1 Voter Behavior

Each voter $i \in \mathcal{N}$ is characterized by a *response function*, $p_i : X^{\mathcal{J}^0} \to \Delta(\mathcal{J})$, where $\Delta(Y)$ denotes the space of all possible lotteries over the set Y. Such a function can be thought of as the strategy of voter i, for example. We will denote the probability a alternative $j \in \mathcal{J}$ receives voter i's vote, conditional on policy proposal vector x, by $p_i(j; x)$.

Throughout the paper, we assume that each p_i characterizes an independent stochastic process.[3] This is stated formally below.

Assumption 1 (Independence) *Conditional on a vector of policy proposals, $x \in Y$, the set of $p_i(x)$ are mutually independent random variables, for all $i \in \mathcal{N}$.*

We now use the set of $p_i(x)$ to define two different candidate objective functions. For simplicity, we define objective functions with respect to pure strategies, as the extension to mixed strategies is obvious.

[3]That is, conditional on the vector of policy proposals x, voters act independently.

2.2 Maximizing Plurality

Given x_{-j}, an *expected plurality maximizing candidate* $j \in \mathcal{J}^0$ seeks to maximize

$$V_j(x) = \frac{1}{N} E \left[\sum_{i=1}^{N} \mathbf{1} \{p_i(x) = j\} - \max_{k \in \mathcal{J}^0, k \neq j} \left[\sum_{i=1}^{N} \mathbf{1} \{p_i(x) = k\} \right] \right]. \tag{2.1}$$

That is, an expected plurality maximizing candidate seeks to maximize the difference between her own vote share and the maximum vote share received by any of the other candidates.[4]

In the case of two candidates without abstention,

$$V_1(x) = \frac{1}{N} \sum_{i=1}^{N} [p_i(1; x) - p_i(2; x)]. \tag{2.2}$$

It should be noted that Equation 2.2 does not directly extend to cases with more than 2 alternatives.

2.3 Maximizing Probability of Victory

$$R_j(x) = \Pr \left[\sum_{i=1}^{N} \mathbf{1} [p_i(x) = j] > \max_{k \neq j} \left[\sum_{i=1}^{N} \mathbf{1} [p_i(x) = k] \right] \right] + \sum_{t=2}^{J^0} \frac{1}{t} \Delta(j, t; p(x)), \tag{2.3}$$

where $\Delta(j, t; p)$ denotes the probability of a tie between t candidates including j, given $p(x) = \{p_1(x), \ldots, p_N(x)\}$.

We will refer to the following lemma in our proof of Theorem 1. It is stated without proof, as it follows immediately from Assumption 1

Lemma 1 *$R_j(x)$ is continuously differentiable in $p(x)$ for all j, for arbitrary values of J.*

3 The Main Result

First it is shown that, for all N with $J = 2$, maximizing expected plurality, $V_j(x)$ is equivalent to maximizing expected vote share, $\frac{1}{N} \sum_{i=1}^{N} p_i(j; x)$.

[4]Note that expected plurality maximizing candidates are assumed to not care about beating alternatives which can not win the election anyway, such as abstention. This implicitly rules out nonstrategic alternatives which can win the election, such as a choice of "None of the above", for instance.

Proposition 1 *Assume that $J = 2$ and the $p_i(x)$ are, conditional on x, mutually independent random variables (i.e., Assumption 1 holds). Then, for any j and all x_{-j},*

$$\arg\max_X V_j(x) = \arg\max_X \frac{1}{N}\sum_{i=1}^{N} p_i(j;x).$$

Proof: By Assumption 1 and $J = 2$,

$$\begin{aligned} V_1(x) &= \frac{1}{N}\sum_{i=1}^{N}[p_i(1;x) - p_i(2;x)] \\ &= \frac{1}{N}\sum_{i=1}^{N}[2p_i(1;x) - 1]. \\ &= \frac{2}{N}[\sum_{i=1}^{N} p_i(1;x)] - 1. \end{aligned}$$

Thus, since the choice of candidate 1 is arbitrary, $V_j(x)$ is an increasing affine transformation of $\frac{1}{N}\sum_{i=1}^{N} p_i(j;x)$, proving the proposition. ∎

3.1 Admissible Games

We now restrict attention to elections which satisfy an admittedly stringent symmetry condition. In particular, we require for all voters to have identical response functions. Formally, we make the following assumption.

Assumption 2 (Symmetry) *For all $i, j \in \mathcal{N}$ and all $x \in Y$,*

$$p_i(x) = p_j(x).$$

We will describe any game with $J = J^0 = 2$ which satisfies Assumptions 1 and 2 as *admissible.*

It is now shown that, in any admissible game, candidates seeking to maximize probability of victory in the case of an anonymous type distribution and plurality rule will seek to maximize expected plurality, regardless of the number of voters.

The following lemma makes the proof almost immediate.

Lemma 2

$$\sum_{c=\lceil \frac{N}{2} \rceil}^{N} \binom{N}{c} \left[p^{c-1}(1-p)^{N-c-1}(c - Np)\right] \geq 0,$$

where the inequality is strict for all $p \in (0,1)$.

Proof: The cases $p = 0$ and $p = 1$ are trivial. Therefore, assume $p \in (0, 1)$. Let $X = X_N$ be a Binomial(N, p) random variable. Define $Z = Z_N = X - Np$ to be the mean zero standardization of X. Note then that

$$\begin{aligned} \sum_{c=0}^{N} \binom{N}{c} \left[p^c (1-p)^{N-c} (c - Np)\right] &= 0, \\ p(1-p) \cdot \sum_{c=0}^{N} \binom{N}{c} \left[p^{c-1} (1-p)^{N-c-1} (c - Np)\right] &= 0. \end{aligned}$$

It follows that

$$p(1-p) \cdot \sum_{c=D}^{N} \binom{N}{c} \left[p^{c-1} (1-p)^{N-c-1} (c - Np)\right] > 0$$

for any $D > 0$, so that

$$\sum_{c=\lceil \frac{N}{2} \rceil}^{N} \binom{N}{c} \left[p^{c-1} (1-p)^{N-c-1} (c - Np)\right] > 0$$

for all $p \in (0, 1)$, completing the proof. ∎

Theorem 1 *For any admissible game,*

$$x_j \in \arg\max_X V(x) \Leftrightarrow x_j \in \arg\max_X R(x).$$

Proof: To show the result, it suffices to show that $R(x)$ is an increasing function of $p(j; x)$.

Lemma 1 ensures that we can differentiate $R(x)$ with respect to $p(j; x)$. For notational ease, let $R_j = R(x)$ and $p_j = p(j; x)$. Doing so and taking the first derivative of R_j with respect to p_j, we obtain

$$\begin{aligned} \frac{\partial R_j}{\partial p_j} &= \sum_{c=\lceil \frac{N}{2} \rceil}^{N} \binom{N}{c} \left[c p_j^{c-1} (1-p_j)^{N-c} - (N-c) p_j^c (1-p_j)^{N-c-1}\right], \\ &= \sum_{c=\lceil \frac{N}{2} \rceil}^{N} \binom{N}{c} \left[p_j^{c-1} (1-p_j)^{N-c-1} \left(c(1-p_j) - (N-c)p_j\right)\right], \\ &= \sum_{c=\lceil \frac{N}{2} \rceil}^{N} \binom{N}{c} \left[p_j^{c-1} (1-p_j)^{N-c-1} (c - Np_j)\right], \qquad (3.1) \\ &> 0, \qquad (3.2) \end{aligned}$$

where the final inequality comes from Lemma 2. Thus, the probability of victory is a strictly increasing function of the expected vote and by Proposition 1, a strictly increasing function of expected plurality. ∎

Theorem 1 only gives sufficient conditions for best response equivalence. We would like to provide necessary conditions as well, but the following examples show that Assumptions 1 and 2, respectively, are not necessary for best response equivalence.

Example 1 Let $X = \{L, R\}$ be a binary policy space, $\mathcal{J} = \mathcal{J}^0 = \{1, 2\}$, and $N = 3$. The voters' response functions are identical, but do not satisfy Assumption 1. In particular, the voters' responses are given by the following rule, where v_i denotes the action of voter i, and v denotes the vector of all v_i.

$$v = \begin{cases} (1,1,1) & \text{if} \quad x = (L, R) \\ \begin{cases} (1,1,1) & \text{with probability } 0.5 \\ (2,2,2) & \text{with probability } 0.5 \end{cases} & \text{if} \quad x = (L, L) \\ \begin{cases} (1,1,1) & \text{with probability } 0.5 \\ (2,2,2) & \text{with probability } 0.5 \end{cases} & \text{if} \quad x = (R, R) \\ (2,2,2) & \text{if} \quad x = (R, L) \end{cases}$$

That is, in all states, the voters vote unanimously for one candidate, and prefer position L.

Regardless of whether a candidate is maximizing expected vote share, expected plurality, or probability of victory, the pure strategy L weakly dominates all other pure and mixed strategies. In fact, L is the unique best response for either candidate to any strategy chosen by her opponent under any of the three objective functions. Thus, best response equivalence holds in this case, even though Assumption 1 does not hold. $\triangle$

Example 2 Let $X = [0, 1]$, $N = 3$, $\mathcal{J} = \mathcal{J}^0 = \{1, 2\}$, and let the voters' response functions satisfy Assumption 1. In particular, assume the following response functions.

$$\begin{aligned} p_1(1; x) &= \frac{1}{2} \\ p_2(1; x) &= \frac{1}{2} + \frac{1}{2}(x_1 - x_2) \\ p_3(1; x) &= \frac{1}{2} - \frac{1}{2}(x_1 - x_2). \end{aligned}$$

Thus, voter 1's behavior is completely unresponsive to the policies announced by the candidates, while voters 2 and 3 each are more likely to choose the candidates announcing the rightmost and leftmost policies, respectively.

It follows easily that the expected vote share and expected plurality of either candidate is invariant to the vector of policies chosen by the candidates, with each candidate receiving an expected vote share of 0.5 and an expected plurality of zero. In calculating the best response correspondence for candidate 2 under maximization of probability of victory, we obtain

$$\begin{aligned} R_2(x) &= (0.5)\left[1 - 2[x_1 - x_2]^2 + 2[x_1 - x_2]^2\right] \\ &= 0.5 \end{aligned}$$

for all choices of x_1 and x_2, implying that a probability of victory maximizing candidate is indifferent between all policies, regardless of the opponent's strategy. Since this holds under all of the three objective functions, best response equivalence holds in this model, in which voters' behavior does not satisfy Assumption 2. △

The above example can be generalized in the following way. This general equivalence will hold in any two candidate election in which the set of voters' response functions satisfy the following condition. Let p denote the set of voters' response functions, and let Φ be an ordering[5] of $\mathcal{N}$. Then both candidates are indifferent as to which policy they choose, regardless of the strategy of the other candidates, under expected vote share, expected plurality, or probability of victory maximization, if, for all $i \in \mathcal{N}$, for some ordering Φ, and for all $x, y \in X$,

$$p_i(x) + p_{\Phi(i)}(x) = p_i(y) + p_{\Phi(i)}(y).$$

Thus, for every voter i whose behavior responds to the policies chosen by the candidates, there exists another voter j whose behavior exactly counterbalances i's behavior.[6]

This section is concluded with an example of another special case when maximizing plurality and maximizing probability of victory yield equivalent best response strategies, regardless of the number of candidates or voters.

Example 3 (Identical Platforms) Let $J = J^0 \geq 2$ and consider candidate 1. Assume that $x_2 = x_3 \ldots = x_J = x^*$. So long as this assumption holds, then maximizing plurality and maximizing probability of winning are equivalent objective functions for candidate 1 (i.e. they yield identical best response functions). The intuition behind this result is straightforward, if not obvious. Any votes lost (alternatively, gained) by candidate 1 as a result of a change in her platform are gained (lost) equally (in expectation) by the other candidates, since the change in the difference of utility between candidate 1 and candidate $j \neq 1$ is identical for all $j \neq 1$. Because of this, the expected vote for candidate 1 is a sufficient statistic for the expected vote of every other candidate. Thus, from the perspective of candidate 1, this situation is identical to the case where $J = 2$.[7]

△

4 Abstention

The previous section provided a theorem which strengthens Hinich's statement that, in two candidate elections without abstention and without coordination by voters, maximizing plurality and probability of victory yielded equivalent strategies in equilibrium.

[5] An ordering of a finite set Z is a one-to-one and onto function from Z into itself.

[6] Oddly enough, this condition is, in some sense, similar to the necessary and sufficient conditions characterized by Plott (1967) for existence of a core in the multidimensional spatial competition model.

[7] Indeed, it is possible to relabel this example so that it is *identical* to the case where $J = 2$ and the voters' errors are biased against candidate 1.

As discussed in the conclusion, the proposition proved here for $J = 2$ is both weaker and stronger than Hinich's original claim, but it is obviously concerned with a very special case, since abstention is generally allowed in most elections, for example. When abstention is allowed, maximizing expected vote is generally not equivalent to maximizing plurality, as we show in Example 6. In this section, we provide an example of a 2 candidate election in which abstention is allowed and voters' behavior satisfies both Assumptions 1 and 2, but maximizing plurality and maximizing probability of victory do not exhibit best response equivalence.

4.1 Probability of Victory Revisited

Ledyard (1984) argues that, asymptotically, maximization of $V(x)$ and $R(x)$ are equivalent when $J^0 = 2$ and $J = 3$. However, Ledyard argues at the limit. When $J = 3$, $J^0 = 2$, and voters' behavior satisfies Assumptions 1 and 2, expected plurality can be expressed as

$$V_j(x) = \frac{p(j;x)}{p(1;x) + p(2;x)} - \frac{p(-j;x)}{p(1;x) + p(2;x)}, \tag{4.1}$$

for $j \in \{1, 2\}$.

The next example, due to John Duggan, highlights why best response equivalence fails to hold in 2 candidate elections with abstention, even when voter behavior satisfies Assumptions 1 and 2.

Example 4 Assume throughout that voter behavior satisfies Assumptions 1 and 2. Let $N = 3$, $J = 3$, $J^0 = 2$, and consider two policy positions, $x, y \in Y$, with $x = (x_1, x_2)$ and $y = (x'_1, x_2)$, characterized by the following voter behavior, where $p(z) = (a, b, c)$ means that, given policy proposal vector z, the probability of any given voter voting for candidate 1 is a, while the probability of voting for candidate 2 is b and the probability of abstention is c:

$$\begin{aligned} p(x) &= (0.02, 0.08, 0.90) \text{ and} \\ p(y) &= (0.53, 0.47, 0.00). \end{aligned}$$

We will focus on candidate 1. It is straightforward to compute the following:

$$\begin{aligned} V_1(x) &= 0.05993\bar{3} \\ R_1(x) &= 0.581396, \end{aligned}$$

while

$$\begin{aligned} V_1(y) &= 0.06 \\ R_1(y) &= 0.544946, \end{aligned}$$

so that $V_1(x) < V_1(y)$, but $R_1(x) > R_1(y)$.

The reason that the two objective functions are not equivalent is that, conditional on any given voter showing up, the probability of candidate 1 receiving that voter's vote is much higher at x than it is at y. It is interesting to note that if x_1 and x_1' are her only choices, candidate 1 has a strict incentive to reduce expected turnout if she wishes to maximize her probability of victory. $\triangle$

5 Where Do We Stand and Where Should We Go?

This section discusses the tightness of our assumptions, possible extensions of the results, and what this paper contributes to our understanding of electoral incentives.

5.1 Extensions

For over twenty years, the theoretical literature has been largely silent on the implications of the modeler's choice of candidates' objectives. This paper proposes that a re-examination of this silence is necessary.

First, the claim in Hinich (1977) regarding asymptotic equivalence of maximizing expected vote and maximizing probability of victory in two candidate elections is not obvious. For clarity, we quote the claim.

"If voters in a large electorate act independently, the distribution of a candidate's total vote approximates a normal distribution for Bernoulli trials. The mean of this normal distribution is the expected vote. Thus for large electorates, maximizing probability of victory is equivalent to maximizing expected vote, *which is also equivalent to maximizing plurality since everyone votes.*" [Hinich (1977), pp. 212-213, Italics in original.]

Exactly when Hinich's claim holds is an open question. Theorem 1 states that Hinich's claim is correct for finite electorates whenever voters' behavior satisfies Assumptions 1 and 2. We now show, however, that it is not true that best response equivalence holds in all 2 candidate elections without abstention. In particular, we construct an example in which voters' behavior does not satisfy Assumption 2. We also show that best response equivalence does not hold in any finite electorate.

Example 5 Let $J = J^0 = 2$ and $N = 3$. Consider an election in which $X = \{L, R\}$ and voter behavior is given by

$$\begin{aligned} p_{11}(L,L) = p_{21}(L,L) = p_{31}(L,L) = \tfrac{1}{2} \\ p_{11}(R,R) = p_{21}(R,R) = p_{31}(R,R) = \tfrac{1}{2}, \end{aligned}$$

while

$$\begin{aligned} p_{11}(R,L) &= \varepsilon \\ p_{21}(R,L) &= \frac{3}{4} - \varepsilon \\ p_{31}(R,L) &= \frac{3}{4} - \varepsilon, \end{aligned}$$

and

$$\begin{aligned} p_{11}(L,R) &= 1 - \varepsilon \\ p_{21}(L,R) &= \frac{1}{4} + \varepsilon \\ p_{31}(L,R) &= \frac{1}{4} + \varepsilon \end{aligned}$$

If candidate 1 deviates to y, both her expected vote and expected plurality decrease by ε, but for sufficiently small $\varepsilon > 0$, her probability of victory is

$$\lim_{\varepsilon \to 0} \left[\frac{9}{16} - \frac{9}{8}\varepsilon + 2\varepsilon^2 - 2\varepsilon^3 \right] \quad > \quad \frac{1}{2}.$$

Thus, even if we assume that (x, x) constitutes a strict global Nash equilibrium under maximization of expected vote share, it is not necessarily a Nash equilibrium under maximization of probability of victory. Thus, best response equivalence between the two objective functions does not hold in general two candidate elections.[8]

This example can be extended to arbitrary numbers of voters. Assuming that $J = 2$ and N odd, fix any symmetric strategy profile, x, so that $p_{i1} = \frac{1}{2}$ for all i, and take voter behavior to be such that

$$\begin{aligned} p_{i1}(R,L) &= \varepsilon & \forall i \leq \tfrac{N-1}{2}, \\ p_{i1}(R,L) &= \tfrac{N}{N+1} - \varepsilon & \forall i > \tfrac{N-1}{2}. \end{aligned}$$

Again, deviating from (L, L) to (R, L) decreases both candidate 1's expected vote share and expected plurality by ε, but increases her probability of victory. Indeed, denoting candidate 1's probability of victory by $R(\varepsilon, N)$, it can be shown that

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} R(\varepsilon, N) = \frac{1}{\sqrt{e}} > \frac{1}{2}.$$

Thus, *even asymptotically*, best response equivalence between the two objective functions does not hold in general when voters' behavior fails to satisfy Assumption 2. △

[8] Since this example holds for an open interval of ε, it immediately follows that the example includes an open set of voter behavior.

5.1.1 Multiple Candidates

Another open question regards multiple candidates. What can we say, if anything, about the relationship between different incentives in such a framework? The next example, due to Tom Palfrey, shows that asymptotic equivalence in our framework is not possible without more restrictions.

Example 6 (Losing the Forest for the Trees) This example utilizes the fact that an increase in one candidate's expected vote share does not necessarily imply a decrease in every other candidate's expected vote share.

Let the policy space be the unit interval, voters' preferences be Euclidean, and let there be three candidates. We assume that voter behavior is sincere: voters vote for the candidate whose announced position is closest to the voter's ideal policy. We also assume there is a continuum of voters. Assume that candidates 1 and 2, proposing x and y, respectively, are adopting identical strategies. Candidate 3 is adopting a strategy, z, which is different from that chosen by candidates 1 and 2. As it stands now, candidates 1 and 2 are each receiving 45% of the vote, while candidate 3 is receiving only 10% of the vote. The probability of victory for candidates 1 and 2 is also equal at this strategy profile. In particular, each candidate wins half of the time, while candidate 3 never wins.

Assume now that candidate 1 seeks to maximize her expected vote share and that there exists a policy x' such that, given y and z, she will receive 475% of the vote, and candidate 2 will receive 48% of the vote. Thus, candidate 1's vote share has increased, but her probability of victory has gone to zero, since candidate 2's vote share is higher than candidate 1's.

This example shows that, even with a continuum of voters, a candidate may increase her expected vote share but decrease her probability of victory. This is generally the case when there are more than 2 candidates. △

Notice that Example 6 does not violate equivalence between expected plurality, as defined above, and probability of victory. The question of asymptotic equivalence between expected plurality and probability of victory with more than two candidates is an open question.

A related question when studying elections with more than two candidates regards the definition of expected plurality. For several reasons, the definition given in equation (2.1) is not without problems. First of all, the idea of plurality is not clear in multiple candidate elections. Why should the emphasis be placed solely on the different between the two top vote-getters? Second, the calculation of equation (2.1) when $J \geq 3$ is by no means simple. One might conjecture that we can approximate equation (2.1) by

$$W_j(x) = p(j;x) - \max_{k \in J_0 k \neq j} [p(k;x)].$$

The following example illustrates the failure of equivalence between that approximation and probability of victory in a case with 3 voters and 3 candidates.

Example 7 As mentioned earlier, the calculation of expected plurality is, in general, a very complicated task when $J > 2$. This objective function depends on the expectation of a rank order statistic generated by a sample of mutually dependent observations.[9] Therefore, one might hope to approximate $V_j(x)$ by

$$\hat{V}_j(x) = p(j;x) - \max_{k \in \mathcal{J}^0; k \neq j} p(k;x).$$

This example shows that this approximation does not capture the behavior of $V(x)$.

Suppose $N = 3$, $J = 3$ without abstention, and, at some policy vector, $x \in Y$, the probabilities of any given voter voting for each candidate are given by:

$$\begin{aligned} p_1(x) &= 0.45 \\ p_2(x) &= 0.4 \\ p_3(x) &= 0.15, \end{aligned}$$

yielding

$$\begin{aligned} R_1(x) &= 0.47925 \\ \hat{V}_1(x) &= 0.05 \\ V_1(x) &= -0.0390002. \end{aligned}$$

Now suppose that there exists a $y \in Y$ such that $y_2 = x_2$, $y_3 = x_3$, and

$$\begin{aligned} p_1(y) &= 0.38 \\ p_2(y) &= 0.31 \\ p_3(y) &= 0.31, \end{aligned}$$

yielding

$$\begin{aligned} R_1(y) &= 0.396492 \\ \hat{V}_1(y) &= 0.07 \\ V_1(y) &= -0.322146. \end{aligned}$$

Notice that $\hat{V}_1(y) > \hat{V}_1(x)$, but $R_1(x) > R_1(y)$ and $V_1(x) > V_1(y)$. This difference between V and $\hat{V}$ is due to the fact that $\hat{V}$ represents only a subset of the information represented by V. This subset is strict whenever $J > 2$.

$\triangle$

[9] While the individual behavior of any given agent is independent of the behavior of every other agent, the sum of votes for any given candidate is dependent on the number of votes received by each other candidate, since N is assumed to be fixed.

While this example shows that equivalence between the approximation of V_j and R_j is not going to hold in arbitrary finite electorates when $|J_0| \geq 3$, it says nothing about equivalence between V_j and R_j in these cases.[10] Thus, two open questions are (1) what kind of equivalence, if any, holds between V_j and R_j in elections with more than 3 candidates and (2) what other candidate objective functions might yield best response functions identical to those yielded by maximization of probability of victory? The answer to (2) would hopefully account for the tractability of the proposed objective function.

5.2 Conclusions

In this paper we have attempted to make two contributions to the formal theory of elections. The first of these is to point out that a rigorous statement and proof of Hinich's (1977) claim that, asymptotically, maximizing plurality and maximizing probability of victory yield equivalent strategies in equilibrium in two candidate elections without voter abstention is not as obvious as might have been assumed. This is important if only because the claim has been widely cited in the literature. We also provide a counterexample to the claim in order to show the need for further investigation into the topic.

The second contribution concerns two candidate elections. It is shown in Theorem 1 that, regardless of the number of voters, maximization of plurality and maximization of probability of victory are *equivalent* objective functions (i.e. they yield identical best response correspondences) in two candidate elections without abstention when voters' behavior satisfies Assumptions 1 and 2.

As stated earlier, Theorem 1 is in some respects weaker, and in others stronger, than Hinich's original statement. Hinich's claim does not require our symmetry condition, Assumption 2. On the other hand, Hinich's claim is asymptotic, while Theorem 1 states that the best response functions are identical for any number of voters.

[10] Indeed, a candidate maximizing V_j, rather than the approximation, would not move as the deviator did in this example.

Appendix A A Probabilistic Voting Framework

This appendix lays out a probabilistic voting framework which might generate the p functions used in the paper to describe voter behavior.

Let $X \subset \mathbf{R}^M$ denote the policy space and $\mathcal{N} = 1, 2, \ldots, N$ denote the set of voters. The space of possible types of voter i is T_i, a subset of some finite dimensional Euclidean space, endowed with the usual topology, denoted by $\mathcal{B}$. We will denote the Cartesian product of the N type spaces by T.

The set of alternatives (which may include abstention) is denoted by $\mathcal{J} = 1, 2, \ldots, J$, with $J < N$. We will denote the set of candidates (*i.e.* excluding abstention) by $\mathcal{J}^0 \subset \mathcal{J}$. This set is exogenously given.[11]

For any voter i, t_i is assumed to be distributed according to a probability measure, $\Phi_i : T_i \to [0, 1]$. Voter i is assumed to observe her type and the policy announcement, $x \in X^{J^0}$. We assume the existence of a function $s_i : T_i \times X^{J^0} \to \mathcal{J}$, mapping policy announcements and types into i's vote choice. We also assume that, for all i, Φ_i and s_i satisfy the following properties:

Assumption 3 (A1) *Φ_i is continuously differentiable in t_i and s_i is continuously differentiable in t_i and x.*

Assumption 4 (A2) *For all $b \in \mathcal{B}$, $\Phi(\mathcal{B}) > 0$.*

Assumption 5 (A3) *For all $j \in \mathcal{J}^0$ and all $x \in X^{J_0}$, there exists $b \in \mathcal{B}$ such that $t_i \in b$ implies $s_i(x, t_i) = j$.*

Assumption A1 ensures that the probability that a randomly drawn voter votes for any given candidate is well behaved. Assumptions A2 and A3 together ensure that the expected vote for any candidate $j \in \mathcal{J}^0$ is strictly positive for all policy positions (this is shown formally, below, in Lemma 3). Any Φ_i and s_i satisfying Assumptions A1, A2, and A3 will be referred to as *admissible*.One final assumption is

Assumption 6 (A4) *For any pair of voters i, j with $i \neq j$, $s_i = s_j$, $\Phi_i = \Phi_j$ and t_i, t_j are distributed independently of each other. That is, given some $N < \infty$, let $\tau = (t_1, \ldots, t_N)$ be distributed according to $F^N(\tau) = \prod_{i=1}^N \Phi(t_i)$.*

Finally, any F for which there exists an admissible Φ such that F satisfies A4 will also be referred to as admissible.

Candidates are assumed to announce policy positions, $x_j \in X$, with $x = (x_1, \ldots, x_{J^0})$ with no abstention. A candidate's announcement is assumed to be implemented if she

[11] That is, we do not allow for strategic candidate entry.

wins the election.[12] Two objective functions are examined: (1) maximizing probability of victory and (2) maximizing expected plurality. We assume that s, the individual strategy for each voter, if common knowledge. Candidates do not observe the type profile of the voters, τ, but do possess a common prior over voter types which is equal to F. Given a vector of candidate actions, x, candidate j perceives the election outcome as a random variable which is determined by the realization of voter types. Thus, the framework defines a Bayesian extensive form game.

A.1 Voter Behavior

Given a policy profile x, let $p_i(j;x)$ denote the probability that voter i chooses candidate j, or, by the assumption that strategies are symmetric with respect to types and types are *i.i.d.*,

$$p(j;x) = \Pr\left[\{t \in T_i | s(x,t) = j\}\right]. \tag{A.1}$$

Denote the vector of $p(j;x)$ for all j by $p(x)$ and the vector of $p(k;x)$ for all $k \neq j$ by $p(-j;x)$.

The following lemmata follow immediately.

Lemma 3 *For any admissible Φ and any $x \in Y$, $p(x) \in$ Int(Δ^J).*

Proof: Take any $x \in Y$ and any candidate j. Then, by Assumption A3, x and j define a $b_{j,x} \in \mathcal{B}$ such that for all $t_i \in b_{j,x}$ $s(x,t_i) = j$. Assumption A2 implies that the $\Phi(b) > 0$, establishing the lemma. ∎

Lemma 4 *For any admissible Φ, $p(x)$ is everywhere continuously differentiable.*

Proof: This follows immediately from Assumption A1. ∎

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